1. *Define the group $Inn(G)$ of inner automorphisms of a group G. Show that $Inn(G) \cong G/Z(G)$, where $Z(G)$ is the center of G.*

   Solution: See Theorem 9.4 from the book "Contemporary Abstract algebra" by Joseph A. Gallian.

2. *Describe the automorphism group of a cycle group of order n.*
   Solution: See theorem 6.5 from the book "Contemporary Abstract algebra" by Joseph A. Gallian.

3. *Let G has even order $2n$. Suppose that exactly half of the elements of G have order 2 and the rest form a subgroup H of order n. Prove that $o(H)$ is odd and that H is abelian.*
   Solution: The order of H must be odd otherwise H contains an element of order 2 using Cauchy's theorem.

   The index of the subgroup H in G is 2, hence H is a normal subgroup.

   So it remains to prove that H is abelian.

   Let $a \in G \setminus H$ be an element of order 2. As $a \notin H$, the left coset aH is different from H. Since the index of H is 2, we have $aH = G \setminus H$. So for any $h \in H$, the order of ah is 2.

   It follows that we have for any $h \in H$,

   $$e = (ah)^2 = ahah,$$

   where e is the identity element in G.

   Equivalently, we have

   $$aha^{-1} = h^{-1}$$

   for any $h \in H$. (Remark that $a = a^{-1}$ as the order of a is 2.)

   Using this relation, for any $h, k \in H$, we have $(hk)^{-1} = a(hk)a^{-1} = (aha^{-1})(aka^{-1}) = h^{-1}k^{-1} = (kh)^{-1}$. As a result, we obtain $hk = kh$ for any $h, k \in H$. Hence the subgroup H is abelian.

4. *Let G be a group and let H be a subgroup of G. Let G acts by left multiplication on the set G/H of all left cosets of H in G. Let ψ_H denotes the associated permutation representation of this action. Show that*
   (a) The action of G on G/H is transitive.
   (b) Stabiliser of the left coset $eH = H$ is the subgroup H.
   (c) Kernel of ψ_H is the largest normal subgroup of G contained in H.

   Solution: Let us denote G/H by A and let the action $\psi_H : G \to S_A$ is defined by $\psi_H(g)(g'H) := gg'H$ where S_A is the set of all permutations on A.
   (a) Let us take g_1H and g_2H be arbitrary two elements of G/H.

   Then, clearly $\psi_H(g_2g_1^{-1})(g_1H) = g_2H$ which shows that the action is transitive.
   (b) Stabilizer of $eH = H$ in G is $G_H := \{g \in G : \psi_H(g)H = H\} = \{g \in G : gH = H\} = H$.
   (c) First we will show that $ker(\psi_H) \subseteq H$. Let $g \in ker(\psi_H) \implies \psi_H(g)(g'H) = g'H$ for all $g' \in G$. Take, $g' = e$ and so we get $g \in H$ i.e. $ker(\psi_H) \subseteq H$.
   Therefore, $ker(\psi_H)$ is a normal subgroup in H. Now, let $K \subseteq H$ is a normal subgroup of G i.e. $g^{-1}Kg = K$ for all $g \in G$. Let $k \in K$ and $g \in G$ be an arbitrary. Then $g^{-1}kg \in K \implies g^{-1}kg \in H \implies kgH = gH \implies \psi_H(k)(gH) = gH$ and we are done.

5. *Show that if G is a group of order p^n, where p is a prime and n is a positive integer, then every subgroup of G of index p is normal in G (Hint: Use the action in problem 4).*
Solution:

Let H be a subgroup of G such that $[G : H] = p$. Consider the action of G on G/H which induces the permutation representation $\psi_H : G \to S_p$. We know from previous problem that $K := ker(\psi_H) \subseteq H$. Then, G/K is isomorphic to a subgroup of S_p, and so $|G/K|$ will divide the order of S_p i.e. $p!$. But it must also divides $|G| = p^n$ and hence, it follows that $|G/K| = p$. Since $|G/K| = [G : K] = [G : H][H : K] = p[H : K]$, it follows that $[H : K] = 1$ i.e. $K = H$ and we are done.